LATTICES OF A LIE GROUP AND SEIFERT FIBRATIONS

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§1. Introduction

Let L be a Lie group with finitely many components, K a maximal compact subgroup of L, and S a connected closed normal subgroup of L. Then KS is closed, and we have a fiber bundle

$$K\backslash KS \to K\backslash L \to KS\backslash L.$$

L acts on $K\backslash L$ by right multiplication. L acts also on $KS\backslash L$ by right multiplication; let N denote the kernel of this action, i.e., $N = \{ g \in L;\ KSxg = KSx \text{ for all } x \in L \}$. The action of N on $K\backslash L$ leaves all fibers invariant; in other words, we have a family of right N-spaces parametrized over $KS\backslash L$.

Lemma 1. <u>The right N-spaces $K\backslash KSx$ $(x \in L)$ are equivalent</u>.

Proof. Since K is compact, $K\backslash L$ has an L-invariant Riemannian metric. Fix such a Riemannian metric. Pick two distinct fibers $K\backslash KSx$ and $K\backslash KSy$ $(x, y \in L)$. It suffices to construct an N-equivariant diffeomorphism from $K\backslash KSx$ onto $K\backslash KSy$ when they are sufficiently close to each other, because $KS\backslash L$ is connected.

Fix a point p of $K\backslash KSx$ and let d be the distance between p and $K\backslash KSy$. $K\backslash L$ is complete and $K\backslash KSy$ is closed; therefore, d is positive and can be achieved as the length of a geodesic γ connecting p and a point q of $K\backslash KSy$. S is contained in N and acts transitively on each fiber. The action of an element s of S sends γ to a geodesic $\gamma \cdot s$ of the same length d connecting $p \cdot s$ and $q \cdot s$. Thus the distance from a point of $K\backslash KSx$ to $K\backslash KSy$ is independent of the choice of the point, and γ is one of the shortest geodesic connecting $K\backslash KSx$ and $K\backslash KSy$. Therefore γ is perpendicular to $K\backslash KSx$ at p. Let $(T_p(K\backslash KSx))^{\perp}$ denote the orthogonal complement of the tangent space $T_p(K\backslash KSx)$ of $K\backslash KSx$ at p in the tangent space of $K\backslash L$ at p. As the exponential map Exp is a diffeomorphism near the origin, any fiber $K\backslash KSz$ that meets Exp(V) meets Exp(V) exactly once, where V is a sufficiently small neighborhood in $(T_p(K\backslash KSx))^{\perp}$ of the origin. This implies that γ is the unique geodesic of length d connecting p and $K\backslash KSy$, as long as $K\backslash KSy$ is sufficiently close to $K\backslash KSx$. Let us suppose that this is the case. Then the correspondence $p \cdot s \mapsto q \cdot s$ $(s \in S)$ defines a diffeomorphism $K\backslash KSx \to K\backslash KSy$, which is obviously N-equivariant because it sends a point in $K\backslash KSx$ to the unique point closest to it in $K\backslash KSy$ and N acts on $K\backslash L$ by isometries. □

Remark. The N-equivariant diffeomorphism above defines a local trivialization of the fiber bundle $K\backslash L \to KS\backslash L$ so that the action of N on $K\backslash L$ is locally a product of the action of N on a fiber and the action of a trivial group on the base.

If G is a lattice of L, the action of L on $K\backslash L$ restricts to an action of G on $K\backslash L$. $H = G\cap N$ is a normal subgroup of G which leaves the fibers invariant. By lemma 1, we have a fiber bundle:

$$K\backslash KS/H \to K\backslash L/H \to KS\backslash L.$$

The quotient group $\Gamma=G/H$ acts on $K\backslash L/H$ and $KS\backslash L$ such that $(K\backslash L/H)/\Gamma = K\backslash L/G$ and $(KS\backslash L)/\Gamma = KS\backslash L/G$; the fiber bundle map induces a map:

$$q: K\backslash L/G \to (KS\backslash L)/\Gamma.$$

Note that $KS\backslash L$ can be naturally identified with $(S\backslash KS)\backslash(S\backslash L)$, which has an $(S\backslash L)$-invariant (and hence L-invariant) Riemannian metric. Thus Γ can be thought of as a subgroup of the group $I(KS\backslash L)$ of all the isometries of $KS\backslash L$ with respect to this Riemannian metric.

Suppose that Γ is discrete in $I(KS\backslash L)$. Then the isotropy subgroup Γ_v of Γ at $v\in KS\backslash L$ is finite for each v, and the inverse image $q^{-1}([v])$ of the orbit $[v] \in (KS\backslash L)/\Gamma$ is $((K\backslash KSx)/H)/\Gamma_v$, where $v = KSx$ $(x \in L)$. Thus a "fiber" of q is homeomorphic to a quotient of the "general fiber" $K\backslash KS/H$ by an action of a finite group; i.e., q is a Seifert fibration [2].

In this article, we will prove the following structure theorem using a suitable closed connected normal subgroup S.

Theorem 2. *Let L be a non-compact Lie group with finitely many components, K a maximal compact subgroup of L, G a lattice of L. Then there is an orbifold Seifert fibration*

$$K\backslash L/G \rightarrow O^m,$$

<u>where</u> O^m <u>is</u> <u>a</u> <u>Riemannian</u> <u>orbifold</u> <u>of</u> <u>dimension</u> $m > 0$ <u>and</u> <u>of</u> <u>non-positive</u> <u>sectional</u> <u>curvature</u>. <u>If</u> L <u>is</u> <u>amenable</u>, O^m <u>can</u> <u>be</u> <u>chosen</u> <u>to</u> <u>be</u> <u>flat</u>.

The proof will occupy the following two sections. Some secial cases of theorem 2 have been known; see Farrell and Hsiang [3, 4] and Quinn [8]. In §4, we will rationally compute the Wall groups of virtually poly-cyclic groups in terms of certain homology theory using the Seifert fibration.

§2. Non-amenable case

Recall that a Lie group L with finitely many components is <u>amenable</u> if and only if L/R is compact, where R denotes the radical (= the unique maximal connected normal solvable subgroup) of L. See Milnor [6]. In this section we handle the case when L is <u>not</u> amenable. We use R as S, following [4]; i.e., we are going to show that

$$K\backslash L/G \rightarrow KR\backslash L/G$$

is a Seifert fibration with the desired property. As in the previous section, identify $KR\backslash L$ with $(R\backslash KR)\backslash(R\backslash L) = \mathbb{R}^m$ $(m > 0)$. $R\backslash L$ is a non-compact semi-simple Lie group, and $R\backslash KR$ is a maximal compact subgroup of $R\backslash L$. Using the Cartan decomposition and the Killing form, one can introduce an

$(R\backslash L)$-invariant (and hence L-invariant) Riemannian metric g on $\mathbb{R}^m$ with non-positive sectional curvature. In fact, any $(R\backslash L)$-invariant Riemannian metric on $\mathbb{R}^m$ has non-positive sectional curvature. See Helgason [5]. Thus we have a homomorphism $\Phi: L \to I(\mathbb{R}^m, g)$. Let Γ denote the image $\Phi(G)$ of G. To prove the theorem, it suffices to show that Γ is discrete in $I(\mathbb{R}^m, g)$. Let τ denote the natural projection $L \to R\backslash L$. If the image $\tau(G)$ of G in $R\backslash L$ is discrete, then Γ is obviously discrete. Unfortunately $\tau(G)$ may not be discrete in general. We remedy this situation as follows.

Let L_0 denote the identity component of L. $G \cap L_0$ is a subgroup of G with finite index. Therefore it suffices to show that $\Phi(G \cap L_0)$ is discrete in $I(\mathbb{R}^m, g)$. As $(R\backslash (K \cap L_0)R)\backslash (R\backslash L_0)$ can be naturally identified with $\mathbb{R}^m$, we may assume from the beginning that L is connected.

Now there is a semi-simple Lie subgroup S of L such that $L = SR$ and such that $S \cap R$ is discrete (Levi decomposition). Let $\sigma: S \to \mathrm{Aut}(R)$ denote the action of S on R. A sufficient condition for $\tau(G)$ to be discrete in $R\backslash L$ is that the identity component $(\ker \sigma)_0$ of the kernel of σ has no compact factors (Raghunathan[9], p.150). Let C denote the unique maximal compact normal subgroups of $(\ker \sigma)_0$. It is a characteristic subgroup of $(\ker \sigma)_0$, and hence it is normal in $\ker \sigma$ and in S. On the other hand, C commutes with elements of R. Therefore C is normal in L. Let $\pi: L \to L/C$ denote the natural projection, and let $L' = \pi(L)$, $S' = \pi(S)$, $R' = \pi(R)$, $G' = \pi(G)$, $K' = \pi(K)$. Then S is semi-simple, R' is the radical of L', G' is a lattice of L', and K' is a maximal compact subgroup of L'. Let $\sigma':S' \to \mathrm{Aut}R'$ denote the action of S' on R'. Then it is easily observed that $\ker \sigma' = (\ker \sigma)/C$, since $C \cap R$ is finite. So the identity component of $\ker \sigma'$ has no compact factors, and this implies that the image G'' of G' in $R'\backslash L'$ is discrete. Thus the action of G on $\mathbb{R}^m$ factors through a properly discontinuous action of G'' on $K'R'\backslash L = KR\backslash L$. Therefore, Γ is discrete in $I(\mathbb{R}^m, g)$. This completes the proof of theorem 2 when L is not amenable.

Remark. Let $q:K\backslash L/G \to KR\backslash L/G$ be the Seifert fibration constructed above. Then the "fiber" $q^{-1}(KRxG)$ over the point $KRxG \in KR\backslash L/G$ $(x \in L)$ is homeomorphic to

$$(x^{-1}Kx)\backslash(x^{-1}KRx)/(x^{-1}KRx \cap G).$$

It is easily observed that $x^{-1}KRx \cap G$ is a uniform lattice (= discrete cocompact subgroup) of $x^{-1}KRx$. In particular, we have

Corollary 3. Let L be a Lie group with finitely many components, K a maximal compact subgroup of L, R the radical of L, and G a lattice of L. Then $KR \cap G$ is a uniform lattice of KR.

§3. Amenable case

Now let us assume that L is non-compact and amenable. Let K be a maximal compact subgroup and R the radical of L as before. Since L is amenable, L = KR.

We define a sequence $N^{(j)}$ ($j \geq -1$) of closed characteristic subgroups of L as follows:

(1) $N^{(-1)}$ is the radical R,

(2) $N^{(0)}$ is the nil-radical, i.e., the maximal connected normal nilpotent subgroup, of L,

(3) $N^{(j)}$ is the commutator subgroup $[N^{(j-1)},N^{(j-1)}]$ of $N^{(j-1)}$, for $j > 0$.

It may not be so obvious that $N^{(j)}$'s are closed when $j > 0$; in general, the commutator subgroup of a Lie group may not be closed. This will be observed later, and we continue the construction. There exists an integer k such that $N^{(k)} = \{1\}$. Consider the following sequence:

$$L = KN^{(-1)} \supset KN^{(0)} \supset KN^{(1)} \supset \dots \supset KN^{(k)} = K.$$

There exists an integer i (≥ 0) such that

$$L = KN^{(-1)} = KN^{(0)} = \dots = KN^{(i-1)} \neq KN^{(i)},$$

because L is non-compact. Let us write $M = N^{(i-1)}$ and $N = N^{(i)}$. We introduce a flat L-invariant Riemannian metric on $KN\backslash L$.

Let us study the action of L on $KN\backslash L$ defined by right multiplication. An element ky of KM = L ($k \in K$, $y \in M$) acts on an element KNx ($x \in M$) of $KN\backslash L$ as follows:

$$\begin{aligned} KNx \cdot (ky) &= KNxky \\ &= KN(k^{-1}xk)y. \end{aligned}$$

Note that we have $[M,M] \subset N$; we identify the coset space $KN\backslash L$ with the simply-connected abelian Lie group $(K\cap M)N\backslash M = \mathbb{R}^m$ $(m>0)$. Now the induced action of L on $\mathbb{R}^m$ is:

$$(K\cap M)Nx\cdot(ky) = (K\cap M)N(k^{-1}xk)y.$$

The following are easily observed: (1) this action, when restricted to K, defines a homomorphism α: $K \to \mathrm{Aut}(\mathbb{R}^m)$ and its image $\alpha(K)$ lies in the orthogonal group $O(m)$ with respect to some inner product of $\mathbb{R}^m$, and (2) if $k \in K\cap M$, then $(K\cap M)Nx\cdot k = (K\cap M)Nk^{-1}xk = (K\cap M)N(k^{-1}xkx^{-1})x = (K\cap M)Nx$ for $x \in M$, and so $K\cap M$ acts trivially on $\mathbb{R}^m$. Let β: $M \to (K\cap M)N\backslash M$ denote the natural projection. We now define a map Φ: $L = KM \to \alpha(K)\ltimes((K\cap M)N\backslash M) \subset O(m)\ltimes\mathbb{R}^m = I(\mathbb{R}^m)$ by sending ky $(k\in K,\ y\in M)$ to $(\alpha(k),\beta(y)) \in O(m)\ltimes\mathbb{R}^m$. This is a well-defined homomorphism. Here $\ltimes$'s denote the obvious semi-direct products. Let Γ denote the image of G by Φ in $I(\mathbb{R}^m)$.

It remains to observe that $N^{(j)}$'s are closed and that Γ is a discrete subgroup of $I(\mathbb{R}^m)$. To do this we use the following lemma:

Lemma 4. <u>If</u> N <u>is a connected nilpotent Lie group and</u> H <u>is a discrete cocompact subgroup of</u> N, <u>then the commutator subgroup</u> $[N,N]$ <u>is closed in</u> N <u>and</u> $H\cap[N,N]$ <u>is cocompact in</u> $[N,N]$.

Proof: This is well-known if N is simply-connected, so consider the universal cover $p: U \to N$ of N; it can be identified with the natural projection $U \to U/\Pi$, where Π is the kernel of

p. To see that $[N,N]$ is closed in N, it suffices to show that $N/[N,N]$ is Hausdorff. As $p^{-1}([N,N]) = \Pi[U,U]$, we have homeomorphisms:

$$
\begin{aligned}
N/[N,N] &\cong U/\Pi[U,U] \\
&\cong (U/[U,U])/(\Pi[U,U]/[U,U]).
\end{aligned}
$$

Here $U/[U,U]$ is a Lie group, because U is simply-connected and hence its commutator subgroup $[U,U]$ is closed. Note that the preimage $p^{-1}(H)$ of H is discrete and cocompact in U. Since U is simply-connected, $p^{-1}(H)\cap[U,U]$ is cocompact in $[U,U]$. Therefore, the image $p^{-1}(H)[U,U]/[U,U]$ of $p^{-1}(H)$ by projection: $U \to U/[U,U]$ is discrete. As $\Pi \subset p^{-1}(H)$, $\Pi[U,U]/[U,U]$ is also discrete and hence closed in $U/[U,U]$. Therefore $(U/[U,U])/(\Pi[U,U]/[U,U])$ is Hausdorff. This proves the first statement as observed above.

Since we have homeomorphisms:

$$
\begin{aligned}
[N,N]/H\cap[N,N] &\cong \Pi[U,U]/p^{-1}(H)\cap\Pi[U,U] \\
&\cong [U,U]/p^{-1}(H)\cap[U,U],
\end{aligned}
$$

the second statement is obvious. □

Now we prove

Lemma 5. $N^{(j)}$'s <u>are closed subgroups of</u> L, <u>and</u> Γ <u>is a crystallographic subgroup of</u> $I(\mathbb{R}^m)$.

Proof: If $G\cap R$ is cocompact in $R=N^{(-1)}$, then $G\cap N^{(0)}$ is a discrete cocompact subgroup of $N^{(0)}$ and we can apply lemma 4 to prove that $N^{(j)}$'s are closed for $j \geq 1$. Unfortunately, $G \cap R$ may not be cocompact in R, in general. To remedy this situation we introduce a quotient Lie group L' of L as in the previous section. We may assume that L is connected. We have Levi decomposition L = SR, where S is a connected semi-simple (and hence compact) subgroup, R is the radical as above, and the intersection $S \cap R$ is finite. Let $\sigma : S \to \mathrm{Aut}(R)$ denote the action of S on R. The identity component $(\ker \sigma)_0$ of $\ker \sigma$ is a connected compact normal subgroup of L, because it commutes with elements of R. In particular, $(\ker \sigma)_0 \subset \ker \alpha \subset K$. Let $\pi : L \to L/(\ker \sigma)_0$ be the natural map. Now define: $L' = L/(\ker \sigma)_0$, $G' = \pi(G)$, $K' = \pi(K)$, $S' = \pi(S)$, $R' = \pi(R)$. Then G' is a lattice of L', K' is a maximal compact subgroup of L', S' is a semi-simple subgroup of L', R' is the radical of L', and the action $\sigma' : S' \to \mathrm{Aut}(R')$ of S' on R' is almost faithful, i.e., $\ker \sigma'$ is finite.

Let us define a sequence $N'^{(j)}$ $(j \geq -1)$ of characteristic subgroups of L' by:

(1) $N'^{(-1)} = R'$

(2) $N'^{(0)}$ = the nil-radical of L'

(3) $N'^{(j)} = [N'^{(j-1)}, N'^{(j-1)}]$ for $j \geq 1$,

then $G' \cap R'$ and $G' \cap N'^{(0)}$ are cocompact in R' and $N'^{(0)}$ respectively. By successively using lemma 4, we know that all $N'^{(j)}$'s are closed. Note that $\pi|R : R \to R'$ is a finite covering map; this implies that $N^{(j)}$ is the identity component of $(\pi|R)^{-1}(N'^{(j)})$ for each j. Therefore $N^{(j)}$'s are closed in L.

Next, we show that Γ is a discrete cocompact subgroup of $I(\mathbb{R}^m)$. Note that we have

$$L' = K'N'^{(-1)} = K'N'^{(0)} = \dots = K'N'^{(i-1)} \neq K'N'^{(i)}$$

for the same i and that $K'N'\backslash K'M' = KN\backslash KM$, where $M' = N'^{(i-1)}$, $N' = N'^{(i)}$. $G'\cap N'^{(j)}$ is cocompact in $N'^{(j)}$ for all j. In particular $G'\cap M'$ is cocompact in M'. So the image of G' in $M'\backslash L'$ is discrete; furthermore, it is finite, because $M'\backslash L'$ is compact. Looking at the diagram:

$$\begin{tikzcd}
 & G \arrow[r, dashed] \arrow[d, phantom, "\cap"] & \text{finite} \arrow[d, phantom, "\cap"] \\
\pi^{-1}(M') \arrow[r] \arrow[d] & L \arrow[r] \arrow[d, "\pi"] & \pi^{-1}(M')\backslash L \arrow[d, "\cong"] \\
M' \arrow[r] & L' \arrow[r] \arrow[d, phantom, "\cup"] & M'\backslash L' \arrow[d, phantom, "\cup"] \\
 & G' \arrow[r] & \text{finite}
\end{tikzcd}$$

we know that $G\cap\pi^{-1}(M')$ has a finite index in G. So it suffices to show that the image $\Phi(G\cap\pi^{-1}(M'))$ is a discrete cocompact subgroup of $I(\mathbb{R}^m)$. As $\ker\sigma \subset \ker\alpha$, Φ sends elements in $\pi^{-1}(M') = (\ker\sigma)_0M$ to elements in $\mathbb{R}^m \subset I(\mathbb{R}^m)$. Now consider the following commutative diagram:

$$\begin{tikzcd}
 & L \arrow[rr, "\Phi"] \arrow[d, phantom, "\cup"] & & O(m)\ltimes\mathbb{R}^m \arrow[d, phantom, "\cup"] \\
G\cap\pi^{-1}(M') \arrow[r, phantom, "\subset"] \arrow[d, dash] & \pi^{-1}(M')=(\ker\sigma)_0M \arrow[rr, "\Phi"] \arrow[d, dash] & & \mathbb{R}^m=(K\cap M)\backslash M \arrow[d, dash] \\
 {} & {} & & {}
\end{tikzcd}$$

$$\begin{array}{ccccc} \Big\downarrow & & \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle (\pi|M)_*} \\ G'\cap M' & \subset & M' & \xrightarrow{\quad\Phi'\quad} & (K'\cap M')N'\backslash M' \end{array}$$

where Φ' is the natural map and $(\pi|M)_*$ is the map induced by the restriction of π to M, $\pi|M\colon M\to M'$. $K\cap M$ and $K'\cap M'$ are maximal compact subgroups of M and M', respectively, and $\pi(K\cap M) = K'\cap M'$; therefore, $(\pi|M)^{-1}(K'\cap M') = K\cap M$. Using this, it is easily verified that $(\pi|M)^{-1}((K'\cap M')N') = (K\cap M)N$. Therefore $(\pi|M)_*$ is an isomorphism. Since $(G'\cap M')\cap N' = G'\cap N'$ is cocompact in N', $(G'\cap M')\cap(K'\cap M')N'$ is cocompact in $(K'\cap M')N'$; so $\Phi'(G'\cap M')$ is a discrete cocompact subgroup of $(K'\cap M')N'\backslash M'$. Therefore $\Phi(G\cap\pi^{-1}(M'))$ is a discrete cocompact subgroup of $\mathbb{R}^m$ (and hence in $I(\mathbb{R}^m)$). This completes the proof of lemma 5. $\square$

Thus $K\backslash L/G \to KN\backslash L/G$ is a desired Seifert fibration as observed in the first section. This completes the proof of theorem 2.

Remark. A fiber of the Seifert fibration above has the form $K\backslash KNxG/G$, and is homeomorphic to

$$(x^{-1}Kx)\backslash(x^{-1}KNx)/(x^{-1}KNx\cap G).$$

If G is a lattice of L (which is automatically uniform), then $x^{-1}KNx\cap G$ is a uniform lattice of $x^{-1}KNx$.

§4. A rational computation of Wall's L-groups

Let L be an amenable Lie group with finitely many components, K a maximal compact subgroup of L, and G a uniform lattice of L. Such a discrete group G is virtually poly-cyclic [6]. Conversely, any virtually poly-cyclic group can be embedded discretely and cocompactly in some amenable Lie group [1]. In this section we compute rationally the L-groups of G in terms of certain generalized homology of $K\backslash L/G$.

$K\backslash L$ is diffeomorphic to some euclidean space $\mathbb{R}^n$ and the isotropy subgroup $G_y = x^{-1}Kx \cap G$ of G at $y=Kx$ $(x \in L)$ is finite. The action of G on $\mathbb{R}^n$ is free if G is torsion-free; in general, $\mathbb{R}^n/G$ is an orbifold, which is Seifert fibered over some flat orbifold as observed in the previous section.

Let WG be a contractible free G-complex, and p denote the projection:$(\mathbb{R}^n \times WG)/G \to \mathbb{R}^n/G$, where G acts on $\mathbb{R}^n \times WG$ diagonally. The preimage $p^{-1}([y])$ of an orbit $[y] \in \mathbb{R}^n/G$ by p is homeomorphic to WG/G_y, and p is a sort of Seifert fibration. (It is called a "stratified system of fibrations" in [7].)

Let $L^{-\infty}(G)$ denote the limit of Ranicki's lower L-groups $L^{\langle -j \rangle}(\mathbb{Z}G)$ [10]. Modulo 2-torsion, it coincides with Wall's surgery obstruction group. We have a functor $\mathbb{L}^{-\infty}(-)$ from the category of spaces to the category of Ω-spectra such that the homotopy group of $\mathbb{L}^{-\infty}(X)$ is equal to $L_*^{-\infty}(\pi_1 X)$. Applying $\mathbb{L}^{-\infty}(-)$ to each fiber of p, we obtain a sheaf of spectra, denoted $\mathbb{L}^{-\infty}(p)$. F. Quinn defines the homology group $H_*(\mathbb{R}^n/G;\mathbb{L}^{-\infty}(p))$ of $\mathbb{R}^n/G$ with coefficients $\mathbb{L}^{-\infty}(p)$. See [7], [10]. The following is a rational computation of $L_*^{-\infty}(G)$ in terms of this homology.

Theorem 6. <u>Let</u> G <u>be</u> <u>as</u> <u>above</u>, <u>then</u> <u>there</u> <u>is</u> <u>a</u> <u>natural</u> <u>isomorphism</u>

$$H_*(\mathbb{R}^n/G;\mathbb{L}^{-\infty}(p))\otimes\mathbb{Z}[1/2] \to L_*^{-\infty}(G)\otimes\mathbb{Z}[1/2].$$

The map is induced by the following map between stratified systems of fibrations.

$$\begin{tikzcd}
(\mathbb{R}^n\times WG)/G \arrow[r, "\mathrm{id.}"] \arrow[d, "p"'] & (\mathbb{R}^n\times WG)/G \arrow[d] \\
\mathbb{R}^n/G \arrow[r] & \mathrm{pt.}
\end{tikzcd}$$

Note that $(\mathbb{R}^n\times WG)/G = BG$ is a classifying space for G and that $H_*(\mathrm{pt.};\mathbb{L}^{-\infty}(BG\to \mathrm{pt.})) = L_*^{-\infty}(G)$ [10].

It is to be noted that theorem 6 says that the $\mathbb{L}^{-\infty}(p)$ coefficient homology of $\mathbb{R}^n/G$ is independent (modulo 2 torsion) of the action of G on $\mathbb{R}^n$. It is conceivable that the orbifold $\mathbb{R}^n/G$ has a certain strong rigidity.

Proof of theorem 6. The proof is by induction on the dimension n of K\L. Let $q:\mathbb{R}^n/G \to \mathbb{R}^m/\Gamma$ denote the Seifert fibration constructed in §3. Modulo 2-torsion, we have

$$\begin{aligned}
&H_*(\mathbb{R}^n/G;\ \mathbb{L}^{-\infty}(p)) \\
&\quad\cong H_*(\mathbb{R}^m/\Gamma;\ \bigcup_{w\in\mathbb{R}^m/\Gamma} \mathbb{H}(q^{-1}(w));\mathbb{L}^{-\infty}(p|q^{-1}(w))) \\
&\quad\cong H_*(\mathbb{R}^m/\Gamma;\ \bigcup_w \mathbb{L}^{-\infty}((qp)^{-1}(w)))
\end{aligned}$$

$$= H_*(\mathbb{R}^m/\Gamma;\ \mathbb{L}^{-\infty}(qp))$$

by induction hypothesis, where $\mathbb{H}$ denote the homology theory spectrum [ibid.]. We can prove that $H_*(\mathbb{R}^m/\Gamma;\ \mathbb{L}^{-\infty}(qp))\otimes\mathbb{Z}[1/2]$ is naturally isomorphic to $L_*^{-\infty}(G)$ using the proof of the main theorem of [ibid.] with only some obvious modifications, and this completes the proof of theorem 6. □

Corollary 7. (Novikov Conjecture) _Let_ G _be as above, then the assembly map_

$$H_*(BG;\ \mathbb{L}^{-\infty}(1)) \to L_*^{-\infty}(G)$$

is rationally split injective.

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